

Representations of the Lie Superalgebra $\mathfrak{gl}(1|n)$ in a Gel'fand-Zetlin Basis and Wigner Quantum Oscillators

R.C. King[†]

School of Mathematics, University of Southampton,
Southampton SO17 1BJ, U.K.;

N.I. Stoilova[‡] and J. Van der Jeugt[§]

Department of Applied Mathematics and Computer Science, Ghent University,
Krijgslaan 281-S9, B-9000 Gent, Belgium.

Abstract

An explicit construction of all finite-dimensional irreducible representations of the Lie superalgebra $\mathfrak{gl}(1|n)$ in a Gel'fand-Zetlin basis is given. Particular attention is paid to the so-called star type I representations ("unitary representations"), and to a simple class of representations $V(p)$, with p any positive integer. Then, the notion of Wigner Quantum Oscillators (WQOs) is recalled. In these quantum oscillator models, the unitary representations of $\mathfrak{gl}(1|DN)$ are physical state spaces of the N -particle D -dimensional oscillator. So far, physical properties of $\mathfrak{gl}(1|DN)$ WQOs were described only in the so-called Fock spaces $W(p)$, leading to interesting concepts such as non-commutative coordinates and a discrete spatial structure. Here, we describe physical properties of WQOs for other unitary representations, including certain representations $V(p)$ of $\mathfrak{gl}(1|DN)$. These new solutions again have remarkable properties following from the spectrum of the Hamiltonian and of the position, momentum, and angular momentum operators. Formulae are obtained that give the angular momentum content of all the representations $V(p)$ of $\mathfrak{gl}(1|3N)$, associated with the N -particle 3-dimensional WQO. For these representations $V(p)$ we also consider in more detail the spectrum of the position operators and their squares, leading to interesting consequences. In particular, a classical limit of these solutions is obtained, that is in agreement with the correspondence principle.

[†]E-mail: R.C.King@soton.ac.uk

[‡]E-mail: Neli.Stoilova@UGent.be; Permanent address: Institute for Nuclear Research and Nuclear Energy, Boul. Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria

[§]E-mail: Joris.VanderJeugt@UGent.be

1 Introduction

In this paper we construct all finite-dimensional irreducible representations of the general linear Lie superalgebra $\mathfrak{gl}(1|n)$ [1, 2]. To this end we introduce a Gel'fand-Zetlin basis (GZ-basis) for $\mathfrak{gl}(1|n)$ and write down explicit expressions for the transformation of the basis vectors under the action of the algebra generators. This is analogous to the GZ-basis for $\mathfrak{gl}(n|1)$ and the action of its generators given by Palev [3]. The Lie superalgebra $\mathfrak{gl}(1|n)$ is a central extension of the special linear Lie superalgebra $\mathfrak{sl}(1|n)$. Each finite-dimensional irreducible $\mathfrak{gl}(1|n)$ representation remains irreducible when restricted to $\mathfrak{sl}(1|n)$, and each of the resulting finite-dimensional irreducible representations of $\mathfrak{sl}(1|n)$ is either typical or atypical [1, 2].

The motivation for the present work stems from the fact that a set of generating elements of $\mathfrak{sl}(1|n)$ satisfies the compatibility conditions of the so called Wigner Quantum Oscillator (WQO) [4]-[10]. Therefore to investigate the properties of these $\mathfrak{sl}(1|n)$ WQOs we must consider those representations that are of physical relevance, namely the star representations of type I [11]. These include certain representations $W(p)$ [12] having highest weight $\Lambda = (p; 0, 0, \dots, 0)$, that have a Fock space realisation. A study of these in the case $n = 3$ and $n = 3N$ has revealed WQO models that have a finite, equally spaced energy spectrum, with discrete values of measurements of both spatial coordinates and linear momenta, as well as angular momenta [5, 7, 8, 9]. Moreover, the underlying geometry of the model is non-commutative [13, 14], in the sense that coordinate operator components in general do not commute; thus measurements of two different coordinates, say x and y , of a single particle cannot be performed simultaneously. As such, the WQO models are to be compared with other non-commutative models, see for example [15]-[18].

Here the intention is to set up the mathematical machinery enabling these properties to be explored for WQO models based on any $\mathfrak{sl}(1|n)$ star representation of type I. The results confirm that in all such cases there is a finite number of equally spaced energy levels, that spatial coordinates and linear momenta are discretely quantised, and that the model always exhibits non-commutative geometry. A detailed analysis is presented for those WQO models based on the $\mathfrak{sl}(1|n)$ representations $V(p)$ having highest weight $\Lambda = (1; p-1, 0, \dots, 0)$, and also for a representation of highest weight $\Lambda = (2; 1, 0, \dots, 0)$ that belongs to neither class $W(p)$ nor $V(p)$.

The structure of the paper is as follows. In Section 2 we construct all the finite-dimensional irreducible representations of the Lie superalgebra $\mathfrak{gl}(1|n)$, complete with a specification of the GZ basis vectors $|m\rangle$ and the explicit action of a set of $\mathfrak{gl}(1|n)$ generators on these vectors. The section culminates with formulae for the action on an arbitrary vector $|m\rangle$ of all the odd elements, e_{0j} and e_{j0} , of $\mathfrak{gl}(1|n)$. These are of importance in Section 4, where for $n = DN$ these odd elements play the role of certain linear combinations of position and linear momentum operators for the N -particle D -dimensional WQO. Before this, in Section 3, the formulae of Section 2 are applied to the case of the $\mathfrak{gl}(1|n)$ representations $V(p)$ of highest weight $\Lambda = (1; p-1, 0, \dots, 0)$. The GZ-basis and the action of the $\mathfrak{gl}(1|n)$ generators are given explicitly in a rather more succinct notation than was possible in the general case.

Returning to Section 4, it is here that the WQO is introduced. The Hamiltonian \hat{H} of an N -particle D -dimensional harmonic oscillator takes the form

$$\hat{H} = \sum_{\alpha=1}^N \left(\frac{\hat{\mathbf{P}}_{\alpha}^2}{2m} + \frac{m\omega^2}{2} \hat{\mathbf{R}}_{\alpha}^2 \right), \quad (1.1)$$

with $\hat{\mathbf{P}}_{\alpha}$ and $\hat{\mathbf{R}}_{\alpha}$ D -dimensional vector operators corresponding to the momentum and position of the particle α , each of the same mass m and natural frequency ω . This Hamiltonian can be

re-expressed in the form

$$\hat{H} = \frac{\hbar\omega}{DN-1} \sum_{j=1}^{DN} \{A_j^+, A_j^-\}, \quad (1.2)$$

where the operators A_j^\pm are simple linear combinations of $\hat{R}_{\alpha k}$ and $\hat{P}_{\alpha k}$ for some α and k determined by j . With this notation, the WQO requirement that Hamilton's equations and the Heisenberg equations coincide as operator equations leads to compatibility conditions on the operators A_j^+ and A_j^- that have a non-canonical solution allowing them to be identified, as stated above, with the odd generators, e_{j0} and e_{0j} , respectively, of $\mathfrak{gl}(1|n)$. This identification is then exploited to determine the physical properties of WQO models, including their energy spectrum and the eigenvalues of their spatial coordinate operators, as well as their non-vanishing commutator. The determination of the angular momentum content of the particular $\mathfrak{gl}(1|3N)$ modules $V(p)$ is deferred to Section 5, where it is achieved for all N and all p , by means of a generating function derived from Molien's Theorem [19, 20] applied to the case of the embedding of the rotation group $SO(3)$, with Lie algebra $\mathfrak{so}(3)$ in the group $GL(3N)$, with Lie algebra $\mathfrak{gl}(3N)$. The outcome in the cases $N = 1$ and $N = 2$ is given in detail.

Section 6 is concerned with an understanding of the position operator spectrum. To this end, its crucial features are illustrated in the case $n = 2$, both for a single particle two-dimensional ($N = 1, D = 2$) model based on the $\mathfrak{gl}(1|2)$ representations $V(p)$, and for a two-particle one dimensional ($N = 2, D = 1$) model based on the same representations. In both cases a classical limit is recovered by taking $p \rightarrow \infty$ and $\hbar \rightarrow 0$ in such a way that $p\hbar \rightarrow C$, for some constant C .

A final example of a WQO model, based on a representation belonging to neither the set $W(p)$ nor the set $V(p)$, is given for illustrative purposes in Section 7. This time the results of Section 2, that immediately give the allowed energy levels and spatial coordinates, are augmented by the use of the subalgebra chain $\mathfrak{gl}(3N) \rightarrow \mathfrak{gl}(3) \oplus \mathfrak{gl}(N) \rightarrow \mathfrak{so}(3) \oplus \mathfrak{gl}(N) \rightarrow \mathfrak{so}(3)$ to obtain a complete description of the angular momentum content.

We close with a few brief concluding remarks in Section 8.

Following customary usage in the mathematical physics literature we use in this paper the words “module” and “representation” more or less interchangeably, and often refer to “simple module” as “irreducible representation.”

2 The $\mathfrak{gl}(1|n)$ representations

The Lie superalgebra $\mathfrak{gl}(1|n)$ can be defined as the set of all squared $(n+1)$ -dimensional matrices with rows and columns labelled by indices $i, j = 0, 1, \dots, n$. As a basis in $\mathfrak{gl}(1|n)$ we choose the Weyl matrices e_{ij} , $i, j = 0, 1, \dots, n$, where the odd elements are $\{e_{i0}, e_{0i} | i = 1, \dots, n\}$, and the remaining elements are even. The Lie superalgebra bracket is determined by

$$[e_{ij}, e_{kl}] \equiv e_{ij}e_{kl} - (-1)^{\deg(e_{ij})\deg(e_{kl})} e_{kl}e_{ij} = \delta_{jk}e_{il} - (-1)^{\deg(e_{ij})\deg(e_{kl})} \delta_{il}e_{kj}. \quad (2.1)$$

Note that $\mathfrak{gl}(1|n)_0 = \mathfrak{gl}(1) \oplus \mathfrak{gl}(n)$, where $\mathfrak{gl}(1) = \text{span}\{e_{00}\}$ and $\mathfrak{gl}(n) = \text{span}\{e_{ij} | i, j = 1, \dots, n\}$. For elements x of $\mathfrak{gl}(1|n)$, one defines the supertrace as $\text{str}(x) = x_{00} - \sum_{j=1}^n x_{jj}$. The Lie superalgebra $\mathfrak{gl}(1|n)$ is not simple, and one can define the simple superalgebra $\mathfrak{sl}(1|n)$ as the subalgebra consisting of elements with supertrace 0. However, the representation theory of $\mathfrak{gl}(1|n)$ or $\mathfrak{sl}(1|n)$ is essentially the same (the situation is similar as for the classical Lie algebras $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$), and hence we prefer to work with $\mathfrak{gl}(1|n)$. As an ordered basis in the Cartan subalgebra \mathfrak{h} of $\mathfrak{gl}(1|n)$, we choose $e_{00}, e_{11}, \dots, e_{nn}$, and denote by

$$\epsilon, \delta_1, \dots, \delta_n \quad (2.2)$$

the dual basis in the space \mathfrak{h}^* of all linear functionals of \mathfrak{h} .

The finite-dimensional simple modules of $\mathfrak{gl}(1|n)$ are characterized by their highest weight Λ . Often, one writes

$$\Lambda = m_{0,n+1} \epsilon + \sum_{i=1}^n m_{i,n+1} \delta_i. \quad (2.3)$$

Then, the finite-dimensional simple modules $W([m]_{n+1})$ of the Lie superalgebra $\mathfrak{gl}(1|n)$ are in one-to-one correspondence with the set of all complex $n+1$ tuples $[1, 2]$

$$[m]_{n+1} = [m_{0,n+1}, m_{1,n+1}, \dots, m_{n,n+1}], \quad (2.4)$$

for which

$$m_{i,n+1} - m_{j,n+1} \in \mathbb{Z}_+, \quad \forall i \leq j = 1, \dots, n. \quad (2.5)$$

Within a given $\mathfrak{gl}(1|n)$ module $W([m]_{n+1})$ the numbers (2.4) are fixed. The possibility of introducing a Gel'fand-Zetlin basis in any finite-dimensional simple $\mathfrak{gl}(1|n)$ module $W([m]_{n+1})$ stems from the following proposition.

Proposition 1 *Consider the $\mathfrak{gl}(1|n)$ module $W([m]_{n+1})$ as a $\mathfrak{gl}(n)$ module. Then $W([m]_{n+1})$ can be represented as a direct sum of simple $\mathfrak{gl}(n)$ modules,*

$$W([m]_{n+1}) = \sum_i \oplus V_i([m]_n), \quad (2.6)$$

where

I. All $V_i([m]_n)$ carry inequivalent representations of $\mathfrak{gl}(n)$

$$[m]_n = [m_{1n}, m_{2n}, \dots, m_{nn}], \quad m_{in} - m_{i+1,n} \in \mathbb{Z}_+. \quad (2.7)$$

II.

1. $m_{in} - m_{i,n+1} = \theta_i \in \{0, 1\}$, $1 \leq i \leq n$,
 2. if for $k \in \{1, \dots, n\}$ $m_{0,n+1} + m_{k,n+1} = k - 1$, then $\theta_k = 0$.
- (2.8)

The decomposition (2.6) and conditions (2.8) follow from the character formula for simple $\mathfrak{gl}(1|n)$ modules [21, 22]. If for some $k \in \{1, \dots, n\}$ the condition $m_{0,n+1} + m_{k,n+1} = k - 1$ is satisfied, then the representation is *atypical of type k*. Otherwise, it is typical.

A GZ-basis for the $\mathfrak{gl}(n)$ module is well known [23]. A set of basis vectors is given by a triangular array of numbers satisfying an integral property and the “betweenness conditions”:

$$\left(\begin{array}{ccccc} m_{1n} & \cdots & \cdots & m_{n-1,n} & m_{nn} \\ m_{1,n-1} & \cdots & \cdots & m_{n-1,n-1} & \\ \vdots & & \ddots & & \\ m_{11} & & & & \end{array} \right). \quad (2.9)$$

These “betweenness conditions” take the form

$$m_{i,j+1} - m_{ij} \in \mathbb{Z}_+ \text{ and } m_{i,j} - m_{i+1,j+1} \in \mathbb{Z}_+, \quad 1 \leq i \leq j \leq n-1, \quad (2.10)$$

and they automatically imply the integral property $m_{i,j} - m_{i+1,j} \in \mathbb{Z}_+$ for all meaningful values of the indices.

Using this GZ-basis for simple $\mathfrak{gl}(n)$ modules and Proposition 1 we have

Proposition 2 *The set of vectors*

$$|m\rangle = \begin{pmatrix} m_{0,n+1} & m_{1,n+1} & \cdots & m_{n-2,n+1} & m_{n-1,n+1} & m_{n,n+1} \\ & m_{1n} & \cdots & \cdots & m_{n-1,n} & m_{nn} \\ & m_{1,n-1} & \cdots & \cdots & m_{n-1,n-1} & \\ & \vdots & \ddots & & & \\ & m_{11} & & & & \end{pmatrix} \quad (2.11)$$

satisfying the conditions

1. $m_{i,n+1}$ are fixed and $m_{i,n+1} - m_{j,n+1} \in \mathbb{Z}_+$ $1 \leq i \leq j \leq n$,
2. $m_{in} - m_{i,n+1} = \theta_i \in \{0, 1\}$, $1 \leq i \leq n$,
3. if for $k \in \{1, \dots, n\}$ $m_{0,n+1} + m_{k,n+1} = k - 1$, then $\theta_k = 0$,
4. $m_{i,j+1} - m_{ij} \in \mathbb{Z}_+$ and $m_{i,j} - m_{i+1,j+1} \in \mathbb{Z}_+$, $1 \leq i \leq j \leq n - 1$.

constitute a basis in $W([m]_{n+1})$.

We shall refer to the basis (2.11) as the GZ-basis for $\mathfrak{gl}(1|n)$. The purpose is now to give the explicit action of a set of $\mathfrak{gl}(1|n)$ generators on the basis vectors (2.11). For this purpose, denote by $|m\rangle_{\pm ij}$ the pattern obtained from $|m\rangle$ by the replacement $m_{ij} \rightarrow m_{ij} \pm 1$. Then this action is given by:

$$e_{00}|m\rangle = \left(m_{0,n+1} - \sum_{j=1}^n \theta_j \right) |m\rangle; \quad (2.13)$$

$$e_{kk}|m\rangle = \left(\sum_{j=1}^k m_{jk} - \sum_{j=1}^{k-1} m_{j,k-1} \right) |m\rangle, \quad (1 \leq k \leq n); \quad (2.14)$$

$$e_{k-1,k}|m\rangle = \sum_{j=1}^{k-1} \left(- \frac{\prod_{i=1}^k (l_{ik} - l_{j,k-1}) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1} - 1)}{\prod_{i \neq j=1}^{k-1} (l_{i,k-1} - l_{j,k-1}) (l_{i,k-1} - l_{j,k-1} - 1)} \right)^{1/2} |m\rangle_{+j,k-1}, \quad (2 \leq k \leq n); \quad (2.15)$$

$$e_{k,k-1}|m\rangle = \sum_{j=1}^{k-1} \left(- \frac{\prod_{i=1}^k (l_{ik} - l_{j,k-1} + 1) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1})}{\prod_{i \neq j=1}^{k-1} (l_{i,k-1} - l_{j,k-1}) (l_{i,k-1} - l_{j,k-1} + 1)} \right)^{1/2} |m\rangle_{-j,k-1}, \quad (2 \leq k \leq n); \quad (2.16)$$

$$e_{0n}|m\rangle = \sum_{i=1}^n \theta_i (-1)^{\theta_1 + \dots + \theta_{i-1}} (l_{i,n+1} + l_{0,n+1} + 1)^{1/2} \left(\frac{\prod_{k=1}^{n-1} (l_{k,n-1} - l_{i,n+1} - 1)}{\prod_{k \neq i=1}^n (l_{k,n+1} - l_{i,n+1})} \right)^{1/2} |m\rangle_{-in}; \quad (2.17)$$

$$e_{n0}|m\rangle = \sum_{i=1}^n (1 - \theta_i) (-1)^{\theta_1 + \dots + \theta_{i-1}} (l_{i,n+1} + l_{0,n+1} + 1)^{1/2} \times \left(\frac{\prod_{k=1}^{n-1} (l_{k,n-1} - l_{i,n+1} - 1)}{\prod_{k \neq i=1}^n (l_{k,n+1} - l_{i,n+1})} \right)^{1/2} |m\rangle_{+in}. \quad (2.18)$$

In all these formulas $l_{ij} = m_{ij} - i$.

In order to deduce (2.13)-(2.18), we have used the paper of Palev [3] and the isomorphism φ of $\mathfrak{gl}(n|1)$ onto $\mathfrak{gl}(1|n)$:

$$\varphi(E_{n,n+1}) = e_{0n}, \quad \varphi(E_{n+1,n}) = e_{n0}, \quad \varphi(E_{ij}) = e_{ij}, \quad i, j = 1, \dots, n, \quad (2.19)$$

where E_{ij} ($i, j = 1, \dots, n+1$) and e_{ij} ($i, j = 0, 1, \dots, n$) are the Weyl matrices of $\mathfrak{gl}(n|1)$ and $\mathfrak{gl}(1|n)$ respectively. As compared with the $\mathfrak{gl}(n|1)$ case, note that in the right hand side of (2.17)-(2.18) instead of $|m\rangle_{\pm in}$ we have $|m\rangle_{\mp in}$. This is because of the conditions 2 and 3 in Proposition 2. Also the atypicality factor $(l_{i,n+1} + l_{0,n+1} + 1)$ is different.

For our purposes, it is also necessary to know which of these representations are “unitary representations” with respect to some star condition (or Hermiticity condition) on the Lie superalgebra elements. The star condition of relevance is the antilinear anti-involutive mapping determined by

$$e_{ij}^\dagger = e_{ji}. \quad (2.20)$$

The unitary representations are then those for which there exist a positive definite inner product $\langle | \rangle$ (in the representation space W) such that

$$\langle e_{ij}v|w\rangle = \langle v|e_{ji}w\rangle, \quad (2.21)$$

for all $v, w \in W$. These are, in the terminology of [11], the *star type I* representations. They have been classified in general for $\mathfrak{gl}(m|n)$, and for $\mathfrak{gl}(1|n)$ the conclusion is as follows:

Proposition 3 *The representation $W([m]_{n+1})$ is a unitary representation if and only if*

(a) *The highest weight is real and*

$$m_{0,n+1} + m_{n,n+1} - n + 1 > 0. \quad (2.22)$$

In this case, the representation is typical.

(b) *The highest weight is real and there exists a $k \in \{1, 2, \dots, n\}$ such that*

$$m_{0,n+1} + m_{k,n+1} = k - 1, \quad m_{k,n+1} = m_{k+1,n+1} = \dots = m_{n,n+1}. \quad (2.23)$$

In this case, the representation is atypical of type k .

For those familiar with the terminology, note that the representations under (b) essentially correspond to the covariant and contravariant representations of $\mathfrak{gl}(1|n)$ [24]. They could also be labelled by $m_{0,n+1}$ and a partition λ .

The representations $W([m]_{n+1})$ are unitary if the conditions of Proposition 3 are satisfied, for the inner product corresponding to orthonormal GZ basis vectors, i.e.

$$\langle |m'\rangle | |m\rangle \rangle \equiv (m'|m) = \delta_{m,m'}, \quad (2.24)$$

under the action (2.13)-(2.18).

We end this section with a technical result. In principle, it is sufficient to have the action of the $\mathfrak{gl}(1|n)$ generators e_{kk} ($k = 0, \dots, n$), $e_{k-1,k}$, $e_{k,k-1}$ ($k = 1, \dots, n$), and e_{0n} , e_{n0} as given in (2.13)-(2.18), in order to compute the action of any element e_{ij} on the GZ basis vectors (2.11). For our later application, it is useful to know the explicit action of the odd elements e_{0j} and e_{j0} of $\mathfrak{gl}(1|n)$. It is found that this is given by the following complicated formulae:

$$\begin{aligned} e_{0j}|m\rangle &= \sum_{i_n=1}^n \sum_{i_{n-1}=1}^{n-1} \dots \sum_{i_j=1}^j \theta_{i_n} (-1)^{\theta_1 + \dots + \theta_{i_n-1}} (l_{i_n,n+1} + l_{0,n+1} + 1)^{1/2} \\ &\times \prod_{r=j+1}^n S(i_r, i_{r-1}) \left(\frac{\prod_{k \neq i_{r-1}=1}^{r-1} (l_{k,r-1} - l_{i_r,r}) \prod_{k \neq i_r=1}^r (l_{kr} - l_{i_{r-1},r-1} + 1)}{\prod_{k \neq i_r=1}^r (l_{kr} - l_{i_r,r}) \prod_{k \neq i_{r-1}=1}^{r-1} (l_{k,r-1} - l_{i_{r-1},r-1} + 1)} \right)^{1/2} \\ &\times \left(\prod_{k \neq i_n=1}^n \frac{(l_{kn} - l_{i_n,n})}{(l_{k,n+1} - l_{i_n,n+1})} \right)^{1/2} \left(\frac{\prod_{k=1}^{j-1} (l_{k,j-1} - l_{i_j,j})}{\prod_{k \neq i_j=1}^j (l_{kj} - l_{i_j,j})} \right)^{1/2} |m\rangle_{-i_n,n; -i_{n-1},n-1; \dots; -i_j,j} \end{aligned} \quad (2.25)$$

$$\begin{aligned}
e_{j0}|m\rangle &= \sum_{i_n=1}^n \sum_{i_{n-1}=1}^{n-1} \dots \sum_{i_j=1}^j (1 - \theta_{i_n}) (-1)^{\theta_1 + \dots + \theta_{i_n-1}} (l_{i_n, n+1} + l_{0, n+1} + 1)^{1/2} \\
&\times \prod_{r=j+1}^n S(i_r, i_{r-1}) \left(\frac{\prod_{k \neq i_{r-1}=1}^{r-1} (l_{k, r-1} - l_{i_r, r} - 1) \prod_{k \neq i_r=1}^r (l_{kr} - l_{i_{r-1}, r-1})}{\prod_{k \neq i_r=1}^r (l_{kr} - l_{i_r, r}) \prod_{k \neq i_{r-1}=1}^{r-1} (l_{k, r-1} - l_{i_{r-1}, r-1} - 1)} \right)^{1/2} \\
&\times \left(\prod_{k \neq i_n=1}^n \frac{(l_{kn} - l_{i_n, n})}{(l_{k, n+1} - l_{i_n, n+1})} \right)^{1/2} \left(\frac{\prod_{k=1}^{j-1} (l_{k, j-1} - l_{i_j, j} - 1)}{\prod_{k \neq i_j=1}^j (l_{kj} - l_{i_j, j})} \right)^{1/2} |m\rangle_{+i_n, n; +i_{n-1}, n-1; \dots; +i_j, j},
\end{aligned} \tag{2.26}$$

where $j = 1, \dots, n$, each symbol $\pm i_k, k$ attached as a subscript to $|m\rangle$ indicates a replacement $m_{i_k, k} \rightarrow m_{i_k, k} \pm 1$, and

$$S(k, l) = \begin{cases} 1 & \text{for } k \leq l \\ -1 & \text{for } k > l. \end{cases} \tag{2.27}$$

3 A special class of representations

One interesting class of representations of $\mathfrak{gl}(1|n)$ is that with $[m]_{n+1} = [p, 0, \dots, 0]$, i.e. with highest weight $\Lambda = p\epsilon$, where $p \in \mathbb{Z}_+$. These are covariant representations labelled by the partition (p) . The representation space $W([p, 0, \dots, 0])$ is simply denoted by $W(p)$. The GZ basis vectors of $W(p)$ can be denoted by $|p; \varphi_1, \dots, \varphi_n\rangle$, where the relation to the GZ labels is determined by

$$\varphi_i = \sum_{j=1}^i m_{ji} - \sum_{j=1}^{i-1} m_{j, i-1}. \tag{3.1}$$

The constraints for the GZ labels lead to: $\varphi_i \in \{0, 1\}$ and $\sum_{i=1}^n \varphi_i \leq \min(p, n)$. The representations $W(p)$ and the basis vectors $|p; \varphi_1, \dots, \varphi_n\rangle$ have been constructed by means of Fock space techniques, and the action of the $\mathfrak{gl}(1|n)$ generators is very simple, see [12]. This class is referred to as the class of Fock representations.

In this paper, another special class of representations will be of importance, namely those with $[m]_{n+1} = [1, p-1, 0, \dots, 0]$, i.e. with highest weight $\Lambda = \epsilon + (p-1)\delta_1$, where $p \in \mathbb{Z}_+$. These are again covariant irreducible representations, labelled this time by the partition $(1, 1, \dots, 1) = (1^p)$. The representation space $W([1, p-1, 0, \dots, 0])$ will be denoted by $V(p)$. We shall assume that $p > 1$, since for $p = 1$ it is actually a Fock representation. Note that $V(p)$ is atypical of type 2, and by Proposition 3 it is also unitary.

One can now apply the general GZ-basis construction of the previous section to this special class. The GZ basis vectors of $V(p)$ are of the form

$$\begin{pmatrix} 1 & p-1 & 0 & \dots & 0 & 0 \\ & q_n & 0 & \dots & 0 & 0 \\ & q_{n-1} & 0 & \dots & 0 & \\ & \vdots & \ddots & & & \\ & q_2 & 0 & & & \\ & q_1 & & & & \end{pmatrix} \tag{3.2}$$

with $q_n = p - \theta \geq q_{n-1} \geq \dots \geq q_1 \geq 0$, where $\theta = 0, 1$ (note that $\theta = 1 - \theta_1$ in the description of (2.12)). Since most of the entries in the GZ-array are zero, it will be more convenient to denote

this vector by

$$|\theta; q_n, q_{n-1}, \dots, q_1\rangle \quad (q_n = p - \theta \geq q_{n-1} \geq \dots \geq q_1 \geq q_0 \equiv 0). \quad (3.3)$$

The actions (2.13)-(2.18) and (2.25)-(2.26) imply that

$$e_{00}|\theta; q_n, q_{n-1}, \dots, q_1\rangle = \theta|\theta; q_n, q_{n-1}, \dots, q_1\rangle; \quad (3.4)$$

$$e_{kk}|\theta; q_n, q_{n-1}, \dots, q_1\rangle = (q_k - q_{k-1})|\theta; q_n, q_{n-1}, \dots, q_1\rangle, \quad (1 \leq k \leq n); \quad (3.5)$$

$$e_{k,k-1}|\theta; q_n, q_{n-1}, \dots, q_1\rangle = \sqrt{(q_k - q_{k-1} + 1)(q_{k-1} - q_{k-2})} \\ \times |\theta; q_n, q_{n-1}, \dots, q_k, q_{k-1} - 1, \dots, q_1\rangle, \quad (2 \leq k \leq n); \quad (3.6)$$

$$e_{k-1,k}|\theta; q_n, q_{n-1}, \dots, q_1\rangle = \sqrt{(q_k - q_{k-1})(q_{k-1} - q_{k-2} + 1)} \\ \times |\theta; q_n, q_{n-1}, \dots, q_k, q_{k-1} + 1, \dots, q_1\rangle, \quad (2 \leq k \leq n); \quad (3.7)$$

$$e_{k0}|\theta; q_n, q_{n-1}, \dots, q_1\rangle = \theta\sqrt{q_k - q_{k-1} + 1} \\ \times |1 - \theta; q_n + 1, q_{n-1} + 1, \dots, q_k + 1, q_{k-1}, \dots, q_1\rangle, \quad (1 \leq k \leq n); \quad (3.8)$$

$$e_{0k}|\theta; q_n, q_{n-1}, \dots, q_1\rangle = (1 - \theta)\sqrt{q_k - q_{k-1}} \\ \times |1 - \theta; q_n - 1, q_{n-1} - 1, \dots, q_k - 1, q_{k-1}, \dots, q_1\rangle, \quad (1 \leq k \leq n). \quad (3.9)$$

These formulas become even simpler in yet another notation for the basis vectors. Let us put

$$r_n = q_n - q_{n-1}, r_{n-1} = q_{n-1} - q_{n-2}, \dots, r_2 = q_2 - q_1, r_1 = q_1, \quad (3.10)$$

then all r_i are nonnegative integers, with $|r| = r_1 + \dots + r_n = q_n$. For the vectors with $\theta = 1$, $q_n = p - 1$, so $|r| = p - 1$. For the vectors with $\theta = 0$, $q_n = p$, so $|r| = p$. Thus all vectors of $V(p)$ are described by:

$$v(\theta; r) \equiv v(\theta; r_1, r_2, \dots, r_n), \quad \theta \in \{0, 1\}, \quad r_i \in \{0, 1, 2, \dots\}, \quad \text{and } \theta + r_1 + \dots + r_n = p. \quad (3.11)$$

In this notation the highest weight vector is $v(1; p-1, 0, \dots, 0)$. The action of the $\mathfrak{gl}(1|n)$ generators on the new basis (3.11) is now given by:

$$e_{00}v(\theta; r) = \theta v(\theta; r); \quad (3.12)$$

$$e_{kk}v(\theta; r) = r_k v(\theta; r), \quad (1 \leq k \leq n); \quad (3.13)$$

$$e_{k,k-1}v(\theta; r) = \sqrt{r_{k-1}(r_k + 1)} v(\theta; r_1, \dots, r_{k-1} - 1, r_k + 1, \dots, r_n), \quad (2 \leq k \leq n); \quad (3.14)$$

$$e_{k-1,k}v(\theta; r) = \sqrt{(r_{k-1} + 1)r_k} v(\theta; r_1, \dots, r_{k-1} + 1, r_k - 1, \dots, r_n), \quad (2 \leq k \leq n); \quad (3.15)$$

$$e_{k0}v(\theta; r) = \theta\sqrt{r_k + 1} v(1 - \theta; r_1, \dots, r_k + 1, \dots, r_n), \quad (1 \leq k \leq n); \quad (3.16)$$

$$e_{0k}v(\theta; r) = (1 - \theta)\sqrt{r_k} v(1 - \theta; r_1, \dots, r_k - 1, \dots, r_n), \quad (1 \leq k \leq n). \quad (3.17)$$

With respect to the inner product

$$\langle v(\theta; r) | v(\theta'; r') \rangle = \delta_{\theta, \theta'} \delta_{r, r'}, \quad (3.18)$$

the representation $V(p)$ is unitary for the star condition (2.20).

4 The N -particle D -dimensional WQO

4.1 The WQO operators

The rest of this paper is devoted to new solutions of the WQO and their physical interpretation. Let us briefly recall the context of WQOs [5]-[10]. Let \hat{H} be the Hamiltonian of an N -particle D -dimensional harmonic oscillator, that is

$$\hat{H} = \sum_{\alpha=1}^N \left(\frac{\hat{\mathbf{P}}_{\alpha}^2}{2m} + \frac{m\omega^2}{2} \hat{\mathbf{R}}_{\alpha}^2 \right), \quad (4.1)$$

with $\hat{\mathbf{P}}_\alpha$ and $\hat{\mathbf{R}}_\alpha$ D -dimensional vector operators corresponding to the momentum and position of the particle α ($\alpha = 1, 2, \dots, N$), m the mass and ω the frequency of each oscillator. We proceed to view this oscillator as a Wigner quantum system: this means that the canonical commutation relations are not required, but are replaced by compatibility conditions between Hamilton's equations and the Heisenberg equations. In other words, Hamilton's equations

$$\dot{\hat{\mathbf{P}}}_\alpha = -m\omega^2 \hat{\mathbf{R}}_\alpha, \quad \dot{\hat{\mathbf{R}}}_\alpha = \frac{1}{m} \hat{\mathbf{P}}_\alpha \quad \text{for } \alpha = 1, 2, \dots, N, \quad (4.2)$$

and the Heisenberg equations

$$\dot{\hat{\mathbf{P}}}_\alpha = \frac{i}{\hbar} [\hat{H}, \hat{\mathbf{P}}_\alpha], \quad \dot{\hat{\mathbf{R}}}_\alpha = \frac{i}{\hbar} [\hat{H}, \hat{\mathbf{R}}_\alpha] \quad \text{for } \alpha = 1, 2, \dots, N, \quad (4.3)$$

should be identical as operator equations. These compatibility conditions are such that

$$[\hat{H}, \hat{\mathbf{P}}_\alpha] = i\hbar m\omega^2 \hat{\mathbf{R}}_\alpha, \quad [\hat{H}, \hat{\mathbf{R}}_\alpha] = -\frac{i\hbar}{m} \hat{\mathbf{P}}_\alpha \quad \text{for } \alpha = 1, 2, \dots, N. \quad (4.4)$$

To make the connection with $\mathfrak{sl}(1|DN)$ we write the operators $\hat{\mathbf{P}}_\alpha$ and $\hat{\mathbf{R}}_\alpha$ for $\alpha = 1, 2, \dots, N$ in terms of new operators:

$$A_{D(\alpha-1)+k}^\pm = \sqrt{\frac{(DN-1)m\omega}{4\hbar}} \hat{R}_{\alpha k} \pm i\sqrt{\frac{(DN-1)}{4m\omega\hbar}} \hat{P}_{\alpha k}, \quad k = 1, \dots, D. \quad (4.5)$$

The Hamiltonian \hat{H} of (4.1) and the compatibility conditions (4.4) take the form:

$$\hat{H} = \frac{\omega\hbar}{DN-1} \sum_{j=1}^{DN} \{A_j^+, A_j^-\}, \quad (4.6)$$

$$\sum_{j=1}^{DN} [\{A_j^+, A_j^-\}, A_i^\pm] = \mp(DN-1)A_i^\pm, \quad i, j = 1, 2, \dots, DN. \quad (4.7)$$

As a solution to (4.7) one can choose operators A_i^\pm that satisfy the following triple relations:

$$\begin{aligned} \{A_i^+, A_j^-, A_k^+\} &= \delta_{jk} A_i^+ - \delta_{ij} A_k^+, \\ \{A_i^+, A_j^-, A_k^-\} &= -\delta_{ik} A_j^- + \delta_{ij} A_k^-, \\ \{A_i^+, A_j^+\} &= \{A_i^-, A_j^-\} = 0. \end{aligned} \quad (4.8)$$

Proposition 4 *The operators A_j^\pm , for $j = 1, 2, \dots, DN$, are odd elements generating the Lie superalgebra $\mathfrak{sl}(1|DN)$.*

This means that one can identify the operators A_j^+ and A_j^- with the $\mathfrak{sl}(1|DN)$ generators e_{j0} and e_{0j} respectively:

$$A_j^+ = e_{j0}, \quad A_j^- = e_{0j}. \quad (4.9)$$

With this identification it should be noted that

$$\hat{H} = \frac{\hbar\omega}{(DN-1)} \sum_{k=1}^{DN} (e_{00} + e_{kk}) = \frac{\hbar\omega}{(DN-1)} \left(DN e_{00} + \sum_{k=1}^{DN} e_{kk} \right). \quad (4.10)$$

To end this subsection, let us also recall that in the case $D = 3$ it is possible to introduce the angular momentum operator of each particle α . These single particle angular momentum operators $\hat{J}_{\alpha j}$ are defined by [7]

$$\hat{J}_{\alpha j} = -\frac{3N-1}{2\hbar} \sum_{k,l=1}^3 \epsilon_{jkl} \{\hat{R}_{\alpha k}, \hat{P}_{\alpha l}\} \quad \alpha = 1, 2, \dots, N, \quad j = 1, 2, 3, \quad (4.11)$$

and take the following form:

$$\hat{J}_{\alpha j} = -i \sum_{k,l=1}^3 \epsilon_{jkl} \{A_{3(\alpha-1)+k}^+, A_{3(\alpha-1)+l}^-\}. \quad (4.12)$$

In terms of these operators the three components of the total angular momentum operator $\hat{\mathbf{J}}$ are given by

$$\hat{J}_j = \sum_{\alpha=1}^N \hat{J}_{\alpha j}, \quad j = 1, 2, 3. \quad (4.13)$$

It is straightforward to verify that with respect to this choice of angular momentum operator $\hat{\mathbf{J}}$ the operators $\hat{\mathbf{R}}_{\alpha}$, $\hat{\mathbf{P}}_{\alpha}$, $\hat{\mathbf{J}}_{\alpha}$ and $\hat{\mathbf{J}}$ all transform as 3-vectors.

4.2 The WQO representations

The Hilbert space (state space) of the WQO is a representation space W of the Lie superalgebra $\mathfrak{gl}(1|DN)$ in such a way that the “observables” (the position and momentum operators $\hat{R}_{\alpha,k}$ and $\hat{P}_{\alpha,k}$) are Hermitian operators. This means that $(A_j^{\pm})^{\dagger} = A_j^{\mp}$, or $e_{j0}^{\dagger} = e_{0j}$. In other words, the state spaces are unitary representations of $\mathfrak{gl}(1|DN)$, in the sense of Proposition 3. Since all “physical operators” in this WQO model are expressed in terms of the A_j^{\pm} , and their actions are known by (2.25)-(2.26), all the relevant physical properties can in principle be deduced.

In previous papers, an investigation was made of the physical properties of the $\mathfrak{gl}(1|DN)$ solutions in the Fock representation spaces $W(p)$, see [5, 8] for $D = 3$, $N = 1$ (the single particle 3-dimensional WQO) and [7, 9] for $D = 3$ and N arbitrary (the last case corresponding to a superposition of N single particle 3-dimensional WQOs). The most striking properties are as follows: the energy of each particle has at most four different eigenvalues (equidistant energy levels); the geometry is non-commutative, in the sense that coordinate operators do not commute; the position and momentum operators have discrete spectra. In this paper, we shall consider more general solutions corresponding to arbitrary unitary representations of $\mathfrak{gl}(1|DN)$, and to the special class of representations $V(p)$.

Some of these properties will be dealt with in general (arbitrary N and D , arbitrary unitary representations); for some others it will be useful to take particular values of N and/or D , or to restrict oneself to specific classes of unitary representations. In the general case, all unitary representations of $\mathfrak{gl}(1|DN)$ are of relevance. In order to simplify notation, we shall use the abbreviation

$$n = DN \quad (4.14)$$

throughout this section.

4.2.1 Energy spectrum

The Hamiltonian \hat{H} is diagonal in the GZ-basis, i.e. the basis vectors $|m\rangle$ are stationary states of the system. This follows from the form of \hat{H} given in (4.10) and the action of e_{00} and e_{kk} given

in (2.13) and (2.14). In each space $W([m]_{n+1})$ there is a finite number of equally spaced energy levels, with spacing $\hbar\omega$:

$$\hat{H}|m\rangle = E_q|m\rangle \quad (4.15)$$

with

$$E_q = \hbar\omega \left(\frac{nm_{0,n+1} + m_{1,n+1} + \dots + m_{n,n+1}}{n-1} - q \right), \quad (4.16)$$

where $q = \sum_{j=1}^n \theta_j$. For typical representations, q takes the values $0, 1, \dots, n$. For a unitary representation $W([m]_{n+1})$ atypical of type k , q takes the values $0, 1, \dots, k-1$. Recall that in that case, the representation labels satisfy

$$m_{0,n+1} + m_{k,n+1} = k-1, \quad m_{k,n+1} = m_{k+1,n+1} = \dots = m_{n,n+1}. \quad (4.17)$$

So for a typical representation, there are $n+1$ equidistant energy levels; for an atypical representation of type k ($k = 1, 2, \dots, n$) there are k equidistant energy levels. The degeneracy of these levels is high, and can be determined explicitly from the GZ-basis labels, or equivalently from the dimensions of the $\mathfrak{gl}(n)$ representations in the decomposition of $W([m]_{n+1})$.

For example, for the special class of representations $V(p)$, atypical of type 2, there are only two distinct energy levels:

$$\begin{aligned} E_0 &= \hbar\omega \left(\frac{p}{n-1} \right) \text{ with degeneracy } \binom{p+n-1}{n-1}, \\ E_1 &= \hbar\omega \left(\frac{p}{n-1} + 1 \right) \text{ with degeneracy } \binom{p+n-2}{n-1}. \end{aligned}$$

4.2.2 Position and momentum operators

The position operators $\hat{R}_{\alpha k}$ ($\alpha = 1, \dots, N$, $k = 1, \dots, D$) of the oscillating particles do not commute with each other

$$[\hat{R}_{\alpha i}, \hat{R}_{\beta j}] \neq 0 \quad \text{for} \quad \alpha i \neq \beta j. \quad (4.18)$$

Similarly

$$[\hat{P}_{\alpha i}, \hat{P}_{\beta j}] \neq 0 \quad \text{for} \quad \alpha i \neq \beta j. \quad (4.19)$$

These relations imply that the WQO belongs to the class of models of non-commutative quantum oscillators [15]-[18], or to theories with non-commutative geometry [13, 14]. In general, the action of $[\hat{R}_{\alpha i}, \hat{R}_{\beta j}]$ is difficult to describe and interpret, even when acting on GZ basis vectors of particular unitary representations. For example, for the representations $V(p)$, one finds:

$$\begin{aligned} [\hat{R}_{\alpha i}, \hat{R}_{\beta j}]v(\theta; r) &= \frac{\hbar}{(n-1)m\omega} \left((-1)^\theta \sqrt{r_l(r_k+1)} v(\theta; r_1, \dots, r_k+1, \dots, r_l-1, \dots, r_n) \right. \\ &\quad \left. - (-1)^\theta \sqrt{r_k(r_l+1)} v(\theta; r_1, \dots, r_k-1, \dots, r_l+1, \dots, r_n) \right), \end{aligned} \quad (4.20)$$

where $k = D(\alpha-1) + i$ and $l = D(\beta-1) + j$ (and it is assumed that $k < l$).

However, the squares of the components of position and momentum operators commute:

$$[\hat{R}_{\alpha i}^2, \hat{R}_{\beta j}^2] = [\hat{P}_{\alpha i}^2, \hat{P}_{\beta j}^2] = 0 \quad \text{for} \quad \alpha i \neq \beta j. \quad (4.21)$$

Furthermore, the GZ basis states $|m\rangle$ are eigenstates of these operators,

$$\begin{aligned} \hat{R}_{\alpha i}^2|m\rangle &= \frac{\hbar}{(n-1)m\omega} (m_{0,n+1} + \dots + m_{n,n+1} - m_{1,n} - \dots - m_{n,n} \\ &\quad + m_{1,k} + \dots + m_{k,k} - m_{1,k-1} - \dots - m_{k-1,k-1})|m\rangle, \end{aligned} \quad (4.22)$$

where $k = D(\alpha - 1) + i$. Thus the spectrum of the position operator component $\hat{R}_{\alpha i}$ is given by the set of values

$$\pm \sqrt{\frac{\hbar}{(n-1)m\omega} \left(\sum_{j=0}^n m_{j,n+1} - \sum_{j=1}^n m_{j,n} + \sum_{j=1}^k m_{j,k} - \sum_{j=1}^{k-1} m_{j,k-1} \right)}, \quad (4.23)$$

where $k = D(\alpha - 1) + i$ and the internal labels $m_{j,l}$ take all possible values allowed by (2.12). For the special representations $V(p)$, (4.22) becomes

$$\hat{R}_{\alpha i}^2 v(\theta; r) = \frac{\hbar}{(n-1)m\omega} (r_k + \theta) v(\theta; r), \quad k = D(\alpha - 1) + i. \quad (4.24)$$

This gives rise to a simple spectrum of the operators $\hat{R}_{\alpha i}$ in these representations.

5 Angular momentum content for $V(p)$

As mentioned before, the case $D = 3$ allows us to introduce angular momentum operators, see (4.11)-(4.13). It is easy to verify that in general the stationary states $|m\rangle$ are not eigenstates of $\hat{J}_{\alpha 3}$, $\hat{\mathbf{J}}_{\alpha}$, \hat{J}_3 and $\hat{\mathbf{J}}^2$. Thus in order to determine the possible values of the total angular momentum j for the N -particle 3-dimensional WQO, it is best to use group theoretical methods rather than the explicit actions of these operators on basis states. The components \hat{J}_k in (4.13) are the generators of an $\mathfrak{so}(3)$ subalgebra of $\mathfrak{gl}(1|n) = \mathfrak{gl}(1|3N)$, that may be identified by means of the following chain of subalgebras:

$$\mathfrak{gl}(1|3N) \rightarrow \mathfrak{gl}(1) \oplus \mathfrak{gl}(3N) \rightarrow \mathfrak{gl}(1) \oplus \mathfrak{gl}(3) \oplus \mathfrak{gl}(N) \rightarrow \mathfrak{gl}(1) \oplus \mathfrak{so}(3) \oplus \mathfrak{gl}(N) \rightarrow \mathfrak{gl}(1) \oplus \mathfrak{so}(3). \quad (5.1)$$

The first step in the branching rule, from $\mathfrak{gl}(1|3N)$ to $\mathfrak{gl}(1) \oplus \mathfrak{gl}(3N)$, is just determined by the GZ-pattern (2.11), where the $\mathfrak{gl}(1)$ value is given by (2.13). The branching rule for $\mathfrak{gl}(3N) \rightarrow \mathfrak{gl}(3) \oplus \mathfrak{gl}(N)$ required in the second step and that for $\mathfrak{gl}(3) \rightarrow \mathfrak{so}(3)$ required in the third step are both rather well known and have been implemented for example in SCHUR [25]. Since they involve coefficients for which there is no known general formula, we content ourselves with giving the results explicitly just for the special representations $V(p)$.

Before doing this we should remark that the embedding of $\mathfrak{so}(3)$ in $\mathfrak{gl}(3N)$ is such that the defining $3N$ -dimensional representation $V_{\mathfrak{gl}(3N)}^{\{1\}}$ of $\mathfrak{gl}(3N)$ decomposes into a direct sum of N copies of the defining 3-dimensional representation $V_{\mathfrak{so}(3)}^1$ of $\mathfrak{so}(3)$, so that all states are of angular momentum $j = 1$. The p th-fold tensor powers of $V_{\mathfrak{gl}(3N)}^{\{1\}}$ therefore contain only states whose angular momentum j is bounded by p . This includes all covariant irreducible representations $V_{\mathfrak{gl}(3N)}^{\{\pi\}}$ where π is any partition of weight p , including the special case $V_{\mathfrak{gl}(3N)}^{\{p\}}$ of particular interest here.

With respect to the first step of (5.1), the representation $V(p)$ of $\mathfrak{gl}(1|3N)$ decomposes into just two symmetric irreducible representations of $\mathfrak{gl}(3N)$:

$$V(p) \rightarrow V_{\mathfrak{gl}(1)}^0 \otimes V_{\mathfrak{gl}(3N)}^{\{p\}} + V_{\mathfrak{gl}(1)}^1 \otimes V_{\mathfrak{gl}(3N)}^{\{p-1\}}, \quad (5.2)$$

labelled by the partitions $\{p\}$ and $\{p-1\}$ respectively. For symmetric representations labelled by a positive integer p , the branching from $\mathfrak{gl}(3N)$ to $\mathfrak{gl}(3) \oplus \mathfrak{gl}(N)$ is determined by the branching rule [26]:

$$V_{\mathfrak{gl}(3N)}^{\{p\}} \rightarrow \sum_{\lambda, |\lambda|=p} V_{\mathfrak{gl}(3)}^{\{\lambda\}} \otimes V_{\mathfrak{gl}(N)}^{\{\lambda\}}, \quad (5.3)$$

where the sum is over all partitions $\{\lambda\}$ with $\lambda_1 + \lambda_2 + \dots = p$. Furthermore, the labelling of representations of $\mathfrak{gl}(3)$ and $\mathfrak{gl}(N)$ requires the extra condition that the length of $\{\lambda\}$ should be at most $\min(3, N)$.

In the case $N = 1$, this step is absent, and for the branching $\mathfrak{gl}(3) \rightarrow \mathfrak{so}(3)$ we have immediately

$$V_{\mathfrak{gl}(3)}^{\{p\}} \rightarrow V_{\mathfrak{so}(3)}^p + V_{\mathfrak{so}(3)}^{p-2} + \dots + V_{\mathfrak{so}(3)}^{1 \text{ or } 0}. \quad (5.4)$$

Thus, using (5.2), the total angular momentum values are given by $j = 0, 1, \dots, p$, with no multiplicities, so that

$$V(p) \rightarrow \sum_{j=0}^p V_{\mathfrak{so}(3)}^j. \quad (5.5)$$

In the case $N \geq 2$, it might appear that the second step, (5.3), cannot be avoided. However, if our intention is simply to identify all possible angular momentum states j and their multiplicities that arise for $V_{\mathfrak{gl}(3N)}^{\{p\}}$, then this may be accomplished by exploiting some classical results from invariant theory. For this purpose, we shall briefly use the Lie group embedding $GL(3N) \supset SO(3)$ rather than the Lie algebra embedding $\mathfrak{gl}(3N) \supset \mathfrak{so}(3)$. For the current class of representations the branching rules associated with these two embeddings are identical.

For any compact continuous subgroup G of $GL(n)$, with group elements g and Haar measure $d\mu(g)$, Molien's function [19] takes the form [20]

$$M(P) = \int \frac{d\mu(g)}{\det(I - P g)}, \quad (5.6)$$

where I is the unit $n \times n$ matrix. The significance of this function in invariant theory is that when expanded in the form:

$$M(P) = \sum_{p=0}^{\infty} n_p P^p, \quad (5.7)$$

the expansion coefficient n_p is the number of linearly independent invariants of G that are homogeneous polynomials in the matrix elements of g of degree p . Equivalently, n_p is the multiplicity of the trivial 1-dimensional identity representation of G in the restriction of $V_{GL(n)}^{\{p\}}$ to G . More generally [27], if the irreducible representation V_G^j , specified by some label j , has character $\chi^j(g)$, with complex conjugate $\chi^{j*}(g)$, then

$$M^j(P) = \int \frac{\chi^{j*}(g) d\mu(g)}{\det(I - P g)} \quad (5.8)$$

has an expansion of the form

$$M^j(P) = \sum_{p=0}^{\infty} n_{p,j} P^p. \quad (5.9)$$

where each coefficient $n_{p,j}$ is the multiplicity of the irreducible representation V_G^j of G in the restriction of $V_{GL(n)}^{\{p\}}$ to G .

Specialising this to the case $n = 3N$ and $G = SO(3)$ we immediately have a means of calculating the angular momentum content of the representation $V_{GL(3N)}^{\{p\}}$ as has been done in the case $N = 2$ by Raychev *et al* [27]. To this end it should be noted that each element g of $SO(3) \subset GL(3N)$ arises as the N th-fold tensor power of a 3×3 matrix which can itself be diagonalised, with diagonal

elements $(z, 1, z^{-1})$ where $z = \exp(i\phi)$ for some real ϕ . With this parametrisation, taking into account the diagonalisation process, the relevant Haar measure is given by [27]

$$d\mu(g) = \frac{1}{2} (1 - \cos \phi) d\phi, \quad (5.10)$$

with $0 \leq \phi < 2\pi$. Moreover, the diagonal form of g is such that

$$\det(I - P g) = ((1 - P z)(1 - P)(1 - P z^{-1}))^N. \quad (5.11)$$

In addition it is well known that the character of the irreducible representation $V_{SO(3)}^j$ of $SO(3)$ is given by

$$\chi^j(g) = \frac{z^{j+\frac{1}{2}} - z^{-j-\frac{1}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}}, \quad (5.12)$$

where it is to be noticed that $\chi^{j*}(g) = \chi^j(g)$. In our case, only integer (rather than half-integer) values of j arise. This allows us to introduce a generating function for $SO(3)$ characters as follows:

$$\begin{aligned} \text{char}(J, g) &= \sum_{j=0}^{\infty} J^j \chi^j(g) = \sum_{j=0}^{\infty} \frac{(J z)^j z^{\frac{1}{2}} - (J z^{-1})^j z^{-\frac{1}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \\ &= \sum_{j=0}^{\infty} \left(\frac{(J z)^j}{1 - z^{-1}} - \frac{(J z^{-1})^j}{z - 1} \right) \\ &= \frac{1}{(1 - J z)(1 - z^{-1})} - \frac{1}{(1 - J z^{-1})(z - 1)}. \end{aligned} \quad (5.13)$$

Hence, we have

$$\begin{aligned} M(P, J)_N &= \sum_{j=0}^{\infty} M^j(P) J^j = \int \frac{\text{char}(J, g^{-1}) d\mu(g)}{\det(I - P g)} \\ &= \oint_{|z|=1} \left(\frac{(1 - z^{-1})^{-1}}{(1 - J z)} - \frac{(z - 1)^{-1}}{(1 - J z^{-1})} \right) \frac{(1 - \frac{1}{2}(z + z^{-1}))}{((1 - P z)(1 - P)(1 - P z^{-1}))^N} \frac{dz}{4\pi i z}, \end{aligned} \quad (5.14)$$

where the fact that $z = \exp(i\phi)$ has allowed the integral over ϕ to be re-written as an integral around the unit circle in the complex z -plane.

In fact, in view of the rather simple dependence on N , we can go even further and introduce the master generating function:

$$\begin{aligned} M(P, J, \mathcal{N}) &= \sum_{N=0}^{\infty} M(P, J)_N \mathcal{N}^N = \sum_{p, j, N=0}^{\infty} n_{p, j, N} P^p J^j \mathcal{N}^N \\ &= \oint_{|z|=1} \left(\frac{(1 - z^{-1})^{-1}}{(1 - J z)} - \frac{(z - 1)^{-1}}{(1 - J z^{-1})} \right) \left(\frac{(1 - \frac{1}{2}(z + z^{-1}))}{1 - \mathcal{N}/((1 - P z)(1 - P)(1 - P z^{-1}))} \right) \frac{dz}{4\pi i z}, \end{aligned} \quad (5.15)$$

where $n_{p, j, N}$ is the multiplicity of the angular momentum state j in the irreducible representation $V_{GL(3N)}^{\{p\}}$ of $GL(3N)$, or equivalently in the representation $V_{\mathfrak{gl}(3N)}^{\{p\}}$ of $\mathfrak{gl}(3N)$. This integral may be evaluated by determining the residues of the integrand at all poles within the unit circle. The result is

$$M(P, J, \mathcal{N}) = 1 + \frac{1}{2} \frac{\mathcal{N}(1 - J)}{J\mathcal{N} - (1 - P)(1 - JP)(J - P)}$$

$$+ \frac{1}{2} \frac{\mathcal{N}(1+J)}{(J\mathcal{N} - (1-P)(1-JP)(J-P))} \sqrt{\frac{\mathcal{N} - (1-P)^3}{\mathcal{N} - (1-P)(1+P)^2}} \quad (5.16)$$

Although not very illuminating, this formula may readily be used to recover the generating functions $M(P, J)_N$ appropriate to any fixed N by expanding (5.16) in powers of \mathcal{N} . In particular we find:

$$M(P, J)_{N=1} = \frac{1}{(1-JP)(1-P^2)} \quad (5.17)$$

$$M(P, J)_{N=2} = \frac{1+JP^2}{(1-JP)^2(1-P^2)^3} \quad (5.18)$$

$$M(P, J)_{N=3} = \frac{1+3JP^2+P^3-J^2P^3-3JP^4-J^2P^6}{(1-JP)^3(1-P^2)^6} \quad (5.19)$$

$$M(P, J)_{N=4} = \frac{(J^3P^{10}+J^3P^8+6J^2P^8+4J^3P^7-4JP^7+J^3P^6-10J^2P^6+P^6-10JP^4+P^4+J^3P^4-4J^2P^3+4P^3+6JP^2+P^2+1)}{(1-JP)^4(1-P^2)^9} \quad (5.20)$$

Finally, it follows from (5.2) that the required generating function for $\mathfrak{gl}(1|3N)$ is obtained by including an additional factor of $(1+P)$ in the numerator of each of these expressions. For example in the case $N=1$, the $\mathfrak{gl}(1|3)$ generating function is

$$G(P, J)_{N=1} = (1+P)M(P, J)_{N=1} = \frac{1}{(1-JP)(1-P)}, \quad (5.21)$$

whose expansion takes the form

$$G(P, J)_{N=1} = 1 + (1+J)P + (1+J+J^2)P^2 + (1+J+J^2+J^2+J^3)P^3 + \dots \quad (5.22)$$

This corresponds of course to our earlier observation (5.5) that in the $N=1$ case $V(p)$ contains states of angular momentum $j=0, 1, \dots, p$, without multiplicity.

By way of a more interesting example, in the case $N=2$ the complete generating function is (see also [27])

$$G(P, J)_{N=2} = \frac{(1+P)(1+P^2J)}{(1-P^2)^3(1-PJ)^2}. \quad (5.23)$$

In the expansion of $G(P, J)_{N=2}$, the coefficient of P^p yields the angular momentum content of the $\mathfrak{gl}(1|6)$ representation $V(p)$,

$$\begin{aligned} G(P, J)_{N=2} = & 1 + (1+2J)P + (3+3J+3J^2)P^2 + (3+7J+5J^2+4J^3)P^3 \\ & + (6+9J+11J^2+7J^3+5J^4)P^4 \\ & + (6+15J+15J^2+15J^3+9J^4+6J^5)P^5 + \dots \end{aligned} \quad (5.24)$$

For example, for $V(3)$ the possible total angular momentum values are $j=0, 1, 2, 3$ with multiplicities 3, 7, 5 and 4 respectively. Note that in general for $\mathfrak{gl}(1|6)$ the total angular momentum values in $V(p)$ are given, as expected, by $j=0, 1, \dots, p$, but each value appears with a certain multiplicity.

To conclude, the complete angular momentum content (including multiplicities) for all representations $V(p)$ of $\mathfrak{gl}(1|3N)$ is resolved by $G(P, J)_N = (1+P)M(P, J)_N$, where $M(P, J)_N$ is the coefficient of \mathcal{N}^N in the expansion of the master generating function (5.16).

6 Position operator properties and a classical limit for $V(p)$

The special class of representations $V(p)$ not only allows a complete analysis of the angular momentum content, it also allows a deeper analysis of the position and momentum operator properties. Let us consider in more detail the single particle ($N = 1$) D -dimensional WQO in the Hilbert space $V(p)$ as representation space. A set of basis vectors is given by

$$v(\theta; r) \equiv v(\theta; r_1, r_2, \dots, r_D), \quad \theta \in \{0, 1\}, \quad r_i \in \{0, 1, 2, \dots\}, \quad \text{and } \theta + r_1 + \dots + r_D = p. \quad (6.1)$$

There are only two distinct energy levels:

$$\hat{H}v(\theta; r) = \hbar\omega\left(\frac{p}{D-1} + \theta\right)v(\theta; r). \quad (6.2)$$

Furthermore, the squares of the position operator components satisfy:

$$\hat{R}_i^2 v(\theta; r) = \frac{\hbar}{(D-1)m\omega}(r_i + \theta)v(\theta; r). \quad (6.3)$$

Similarly, one has

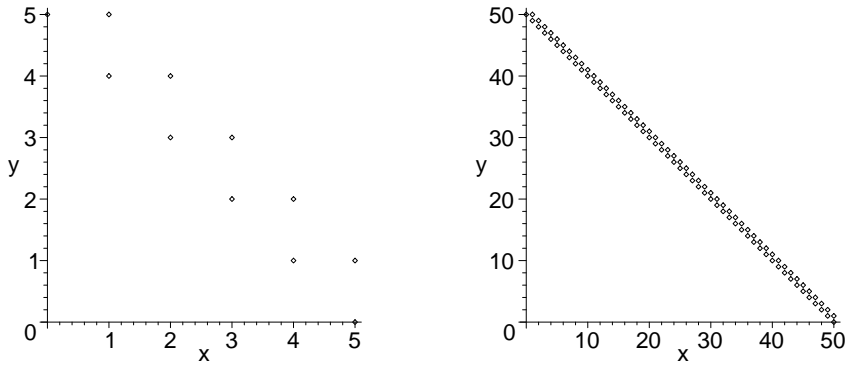
$$\hat{P}_i^2 v(\theta; r) = \frac{\hbar m\omega}{D-1}(r_i + \theta)v(\theta; r). \quad (6.4)$$

In order to illustrate some of the spatial properties, let us first assume that $D = 2$, i.e. we are dealing with a 2-dimensional one particle WQO. The squares of the position operator components commute, $[\hat{R}_1^2, \hat{R}_2^2] = 0$, and the $2p + 1$ basis vectors $v(\theta; r_1, r_2)$ are common eigenvectors:

$$\begin{aligned} \hat{R}_1^2 v(\theta; r) &= \frac{\hbar}{m\omega}(r_1 + \theta)v(\theta; r), & \hat{R}_2^2 v(\theta; r) &= \frac{\hbar}{m\omega}(r_2 + \theta)v(\theta; r), \\ (\hat{R}_1^2 + \hat{R}_2^2)v(\theta; r) &= \frac{\hbar}{m\omega}(p + \theta)v(\theta; r). \end{aligned} \quad (6.5)$$

Let us consider in Figure 1 a plot of the eigenvalues of \hat{R}_1^2 and \hat{R}_2^2 with respect to these eigenvectors, for some values of p .

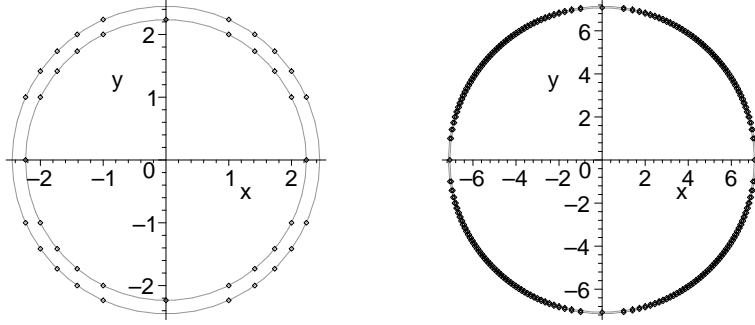
Figure 1: Eigenvalues of \hat{R}_1^2 (on the x -axis) and \hat{R}_2^2 (on the y -axis), for $p = 5$ and $p = 50$ (in units of $\frac{\hbar}{m\omega}$). For each common eigenvector $v(\theta; r_1, r_2)$ the corresponding eigenvalues are plotted.



Since \hat{R}_1 and \hat{R}_2 do not commute, the position of the oscillating particle cannot be determined. However, if the system is in one of the basis states $v(\theta; r_1, r_2)$, the above implies that the particle can

be found in four possible positions, with coordinates $\sqrt{\frac{\hbar}{m\omega}}(\pm\sqrt{r_1+\theta}, \pm\sqrt{r_2+\theta})$ (independent \pm -signs). If we continue in this interpretation, where the positions are determined through measuring the eigenvalues of \hat{R}_1^2 and \hat{R}_2^2 in their common eigenstates, then the possible positions of the particle in the Hilbert space $V(p)$ consists of a collection of such coordinates $\sqrt{\frac{\hbar}{m\omega}}(\pm\sqrt{r_1+\theta}, \pm\sqrt{r_2+\theta})$. This leads to Figure 2. Thus the particle is situated on one of two circles, one with radius $\rho_0 = \sqrt{\frac{p\hbar}{m\omega}}$

Figure 2: Possible positions of the WQO, for $p = 5$ and $p = 50$ (in units of $\sqrt{\frac{\hbar}{m\omega}}$).



and one with radius $\rho_1 = \sqrt{\frac{(p+1)\hbar}{m\omega}}$. The spectrum of the coordinate operators is discrete, however.

The current situation is interesting, since the classical limit of the WQO in this Hilbert space $V(p)$ can be considered. For this purpose, one should examine the situation where the quantum numbers of the system become very large (in the limit to infinity) and \hbar becomes very small (in the limit to 0). So let us deal with the situation where

$$p \rightarrow +\infty, \quad \hbar \rightarrow 0, \quad \text{but} \quad p\hbar \rightarrow C, \quad (6.6)$$

for some constant C . We shall examine what happens to the spectrum of the physical operators \hat{H} , \hat{R}_1 , \hat{R}_2 and $\hat{R}_1^2 + \hat{R}_2^2$ under this limit. Using (6.6), the energy of the system becomes constant, $C\omega$, as the two energy levels coincide under the limiting process. The limit of the spectrum of \hat{R}_1 and \hat{R}_2 is the interval $[-\sqrt{\frac{C}{m\omega}}, \sqrt{\frac{C}{m\omega}}]$. The eigenvalue of $\hat{R}_1^2 + \hat{R}_2^2$ becomes $\frac{C}{m\omega}$, a fixed constant.

So the particle is situated on a circle with radius $\sqrt{\frac{C}{m\omega}}$. In the limit, all possible positions on the circle are possible. This suggests that under this limit the solution of the WQO corresponds to one of the classical solutions

$$x(t) = A \cos(\omega t), \quad y(t) = A \sin(\omega t) \quad (6.7)$$

of a 2-dimensional simple homogeneous harmonic oscillator, determined by the differential equations

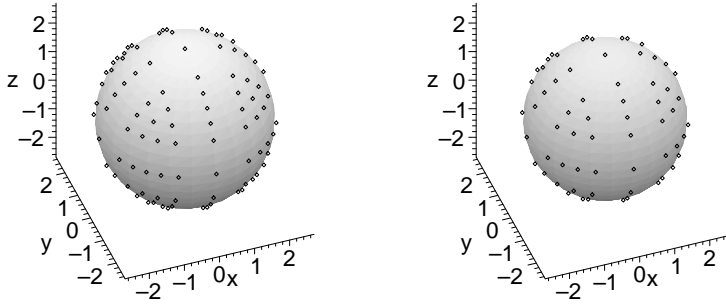
$$\ddot{x}(t) + \omega^2 x(t) = 0, \quad \ddot{y}(t) + \omega^2 y(t) = 0, \quad (6.8)$$

where $\omega^2 = \frac{k}{m}$ (k being the “spring constant”). The classical energy of such a system is $m\omega^2 A^2$, and the classical oscillator moves on a circle with radius A . Under the identification of the constants $C = m\omega A^2$, both the energy and position of the WQO (for this solution) are in agreement with those of the particular solution of the classical oscillator. In other words, the “correspondence principle” holds for this solution of the WQO.

It should be clear that the above considerations about possible particle positions remain valid for arbitrary D . For instance, when $D = 3$, the possible positions of the particle in one of the

basis states of $V(p)$ consists of a collection of coordinates $\sqrt{\frac{\hbar}{m\omega}}(\pm\sqrt{r_1+\theta}, \pm\sqrt{r_2+\theta}, \pm\sqrt{r_3+\theta})$ (independent \pm -signs). The collection of such coordinates leads to Figure 3, i.e. a set of points on two concentric spheres, one with radius $\sqrt{\frac{p\hbar}{2m\omega}}$ and one with radius $\sqrt{\frac{(p+1)\hbar}{2m\omega}}$.

Figure 3: Possible positions of the 3-dimensional WQO, for $p = 5$ (in units of $\sqrt{\frac{\hbar}{2m\omega}}$). In the left figure, the set of points is on a sphere with radius $\sqrt{p+1}$; in the right figure, they are on a sphere with radius \sqrt{p} .



As far as the position or momentum operators for the WQO is concerned, we have not yet considered their time dependence. Following [7]-[9], this is given by

$$\hat{R}_k(t) = \sqrt{\frac{\hbar}{(D-1)m\omega}}(A_k^+ e^{-i\omega t} + A_k^- e^{i\omega t}), \quad \hat{P}_k(t) = -i\sqrt{\frac{m\omega\hbar}{(D-1)}}(A_k^+ e^{-i\omega t} - A_k^- e^{i\omega t}), \quad (6.9)$$

for the D -dimensional single particle case. For a general mixed state of $V(p)$,

$$|x\rangle = \sum_{\theta,r} c_{\theta;r} v(\theta;r), \quad \sum_{\theta,r} c_{\theta;r}^2 = 1, \quad (6.10)$$

one finds that the average value or mean trajectories of the coordinate and momentum operator components is

$$\langle x | \hat{R}_k | x \rangle = 2\sqrt{\frac{\hbar}{(D-1)m\omega}} B_k \cos(\omega t), \quad \langle x | \hat{P}_k | x \rangle = -2\sqrt{\frac{m\omega\hbar}{D-1}} B_k \sin(\omega t), \quad (6.11)$$

where B_k is the constant

$$B_k = \sum_r \sqrt{r_k} c_{0;r_1, \dots, r_D} c_{1;r_1, \dots, r_k-1, \dots, r_D}. \quad (6.12)$$

If the system is in a stationary state (fixed energy eigenvalue), one has either all $c_{0,r} = 0$ or else all $c_{1,r} = 0$. In that case, all B_k are zero. In other words, for stationary states $|x\rangle$ the average values $\langle x | \hat{R}_k | x \rangle$ and $\langle x | \hat{P}_k | x \rangle$ vanish.

The analysis given so far was for a single particle D -dimensional WQO in the representation space $V(p)$ of $\mathfrak{gl}(1|D)$. One can also consider the same representations $V(p)$ of $gl(1|N)$ as state spaces of a system of N 1-dimensional WQO's ($N > 1$, since a single 1-dimensional WQO does not have a solution in terms of $\mathfrak{gl}(1|1)$). In this second interpretation, the basis states

$$v(\theta;r) \equiv v(\theta;r_1, r_2, \dots, r_N), \quad \theta \in \{0, 1\}, \quad r_i \in \{0, 1, 2, \dots\}, \quad \text{and } \theta + r_1 + \dots + r_N = p. \quad (6.13)$$

are the same as in (6.1), and similarly

$$\hat{H}v(\theta; r) = \hbar\omega\left(\frac{p}{N-1} + \theta\right)v(\theta; r), \quad \hat{R}_i^2 v(\theta; r) = \frac{\hbar}{(N-1)m\omega}(r_i + \theta)v(\theta; r). \quad (6.14)$$

In other words, all the previously obtained formulas remain valid, only the interpretation of the system is different: now it consists of N one-dimensional oscillating particles, and \hat{R}_i is the operator corresponding to the position of particle i . In the representation space $V(p)$, the spectrum of the operator \hat{R}_i^2 is $\{0, 1, \dots, p\}$ and that of \hat{R}_i is $\{0, \pm 1, \pm\sqrt{2}, \dots, \pm\sqrt{p}\}$. Let us again assume that $N = 2$, so that we are dealing with just two one-dimensional oscillating particles. If the system is in one of the basis states $v(\theta; r_1, r_2)$, our analysis leads to the interpretation that the position of the first one-dimensional oscillator is $\pm\sqrt{\frac{\hbar(r_1+\theta)}{m\omega}}$, and that of the second one-dimensional oscillator is $\pm\sqrt{\frac{\hbar(r_2+\theta)}{m\omega}}$. Since $\theta + r_1 + r_2 = p$, these positions are not independent but “correlated”. To see this from a different point of view, recall that the positions of the two oscillators cannot be measured simultaneously, since $[\hat{R}_1, \hat{R}_2] \neq 0$. But suppose the system is in an arbitrary state of $V(p)$, and the position of the first oscillator is measured at time $t = 0$. This measurement has one of the eigenvalues $\pm\sqrt{\frac{\hbar r_1}{m\omega}}$ of $\hat{R}_1(t)$ as an outcome, with $r_1 \in \{0, 1, \dots, p\}$. As a consequence of the measurement, the system is then in an eigenstate of \hat{R}_1 for this eigenvalue. It is easy to verify that this eigenstate is unique, given by (we assume $r_1 > 0$)

$$\frac{1}{\sqrt{2}}v(0; r_1, r_2) \pm \frac{1}{\sqrt{2}}v(1; r_1 - 1, r_2), \quad (6.15)$$

with $r_2 = p - r_1$ (the plus sign for the positive eigenvalue, and the minus sign for the negative eigenvalue). The time evolution of this system is then described by

$$|x\rangle = \frac{e^{i\omega t/2}}{\sqrt{2}}v(0; r_1, r_2) \pm \frac{e^{-i\omega t/2}}{\sqrt{2}}v(1; r_1 - 1, r_2). \quad (6.16)$$

Note that $|x\rangle$ is an eigenstate of $\hat{R}_1(t)$ for the eigenvalue $\pm\sqrt{\frac{\hbar r_1}{m\omega}}$. The state (6.16) can be rewritten as follows:

$$\begin{aligned} |x\rangle &= \frac{1}{2\sqrt{2}}[e^{i\omega t/2}v(0; r_1, r_2) + e^{-i\omega t/2}v(1; r_1, r_2 - 1)] \\ &+ \frac{1}{2\sqrt{2}}[e^{i\omega t/2}v(0; r_1, r_2) - e^{-i\omega t/2}v(1; r_1, r_2 - 1)] \\ &\pm \frac{1}{2\sqrt{2}}[e^{i\omega t/2}v(0; r_1 - 1, r_2 + 1) + e^{-i\omega t/2}v(1; r_1 - 1, r_2)] \\ &\mp \frac{1}{2\sqrt{2}}[e^{i\omega t/2}v(0; r_1 - 1, r_2 + 1) - e^{-i\omega t/2}v(1; r_1 - 1, r_2)]. \end{aligned} \quad (6.17)$$

The four linear combinations in square brackets are four eigenvectors of $\hat{R}_2(t)$, with eigenvalues $\sqrt{\frac{\hbar}{m\omega}}\sqrt{p-r_1}$, $-\sqrt{\frac{\hbar}{m\omega}}\sqrt{p-r_1}$, $\sqrt{\frac{\hbar}{m\omega}}\sqrt{p-r_1+1}$ and $-\sqrt{\frac{\hbar}{m\omega}}\sqrt{p-r_1+1}$ respectively. This implies that a measurement of the position of the first oscillator, i.e. of \hat{R}_1 at time $t = 0$ with outcome $\sqrt{\frac{\hbar r_1}{m\omega}}$, imposes strong restrictions on the position eigenvalues of the second oscillator, measured at time t . The two one-dimensional oscillators are not free but correlated in some sense.

Let us also examine the classical limit (6.6) for this second interpretation. In fact, the limit of the spectrum of the operators \hat{H} , \hat{R}_1 , \hat{R}_2 and $\hat{R}_1^2 + \hat{R}_2^2$ is the same as before. So the total energy of the system becomes constant, $C\omega$. The limit of the spectrum of \hat{R}_1 and of \hat{R}_2 is the interval

$[-\sqrt{\frac{C}{m\omega}}, \sqrt{\frac{C}{m\omega}}]$, and the eigenvalue of $\hat{R}_1^2 + \hat{R}_2^2$ is fixed, $\frac{C}{m\omega}$. Thus, under this limit the solution of the 2-particle 1-dimensional WQO corresponds to the classical solution

$$x_1(t) = A \cos(\omega t), \quad x_2(t) = A \sin(\omega t). \quad (6.18)$$

This simply describes a system of two uncoupled identical 1-dimensional harmonic oscillators, but with a fixed phase difference in their oscillations. In other words, the initial conditions of the two oscillators are related: also here there is a correlation. So the ‘‘correspondence principle’’ also holds for this interpretation of the WQO.

7 Analysis for a general example

The results of the previous sections were mainly devoted to the special representations $V(p)$. However, the computations of Section 2 allow us to make an analysis of the WQO properties for any unitary representation. As an example, we consider a representation $W([m]_{3N+1})$ which belongs to neither the set $W(p)$ nor the set $V(p)$, namely the representation corresponding to the $\mathfrak{gl}(1|3N)$ irreducible representation $V_{\mathfrak{gl}(1|3N)}^{\{2,1\}}$ with highest weight $\Lambda = 2\epsilon + \delta_1$. This can be identified with the $\mathfrak{gl}(1|3N)$ module $W([m]_{3N+1})$ with $[m]_{3N+1} = [2, 1, 0, \dots, 0]$.

The branching from $\mathfrak{gl}(1|3N)$ to $\mathfrak{gl}(1) \otimes \mathfrak{gl}(3N)$ is given by

$$V_{\mathfrak{gl}(1|3N)}^{\{2,1\}} \rightarrow V_{\mathfrak{gl}(1)}^2 \times V_{\mathfrak{gl}(3N)}^{\{1\}} + V_{\mathfrak{gl}(1)}^1 \times V_{\mathfrak{gl}(3N)}^{\{2\}} + V_{\mathfrak{gl}(1)}^1 \times V_{\mathfrak{gl}(3N)}^{\{1,1\}} + V_{\mathfrak{gl}(1)}^0 \times V_{\mathfrak{gl}(3N)}^{\{2,1\}}. \quad (7.1)$$

The four terms arising here are associated with the four sets of states:

$$\left(\begin{array}{cccccc} 2 & 1 & 0 & 0 & \cdots & 0 \\ & 1 & 0 & 0 & \cdots & 0 \\ & * & * & 0 & \cdots & \\ & \vdots & \vdots & & \ddots & \\ & * & * & & & \\ & * & & & & \end{array} \right) \quad \left(\begin{array}{cccccc} 2 & 1 & 0 & 0 & \cdots & 0 \\ & 1 & 1 & 0 & \cdots & 0 \\ & * & * & 0 & \cdots & \\ & \vdots & \vdots & & \ddots & \\ & * & * & & & \\ & * & & & & \end{array} \right) \quad \left(\begin{array}{cccccc} 2 & 1 & 0 & 0 & \cdots & 0 \\ & 2 & 0 & 0 & \cdots & 0 \\ & * & * & 0 & \cdots & \\ & \vdots & \vdots & & \ddots & \\ & * & * & & & \\ & * & & & & \end{array} \right) \quad \left(\begin{array}{cccccc} 2 & 1 & 0 & 0 & \cdots & 0 \\ & 2 & 1 & 0 & \cdots & 0 \\ & * & * & 0 & \cdots & \\ & \vdots & \vdots & & \ddots & \\ & * & * & & & \\ & * & & & & \end{array} \right) \quad (7.2)$$

with the entries $*$ taking values determined by the various betweenness conditions. For all states within each of these four sets the corresponding values of $(\theta_1, \theta_2, \dots)$ are given by $(0, 0, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, $(1, 0, 0, \dots, 0)$ and $(1, 1, 0, \dots, 0)$, respectively. In the notation of (4.17) we have

$$E_q = \hbar\omega \left(\frac{6N+1}{3N-1} - q \right) \quad (7.3)$$

with $q = 0, 1, 1$ and 2 , for our four sets of states. It follows that we have just three distinct energy levels with degeneracies as shown below:

$$\begin{aligned} E_0 &= \hbar\omega \frac{6N+1}{3N-1}, & d_0 &= 3N; \\ E_1 &= \hbar\omega \frac{3N+2}{3N-1}, & d_1 &= 9N^2; \\ E_2 &= \hbar\omega \frac{3}{3N-1}, & d_2 &= N(9N^2-1). \end{aligned} \quad (7.4)$$

These correspond precisely to the three $\mathfrak{gl}(1)$ representations, $V_{\mathfrak{gl}(1)}^2$, $V_{\mathfrak{gl}(1)}^1$ and $V_{\mathfrak{gl}(1)}^0$ appearing in (7.1). The degeneracies, d_q have been obtained by using N -dependent formulae for the dimensions of the irreducible representations of $\mathfrak{gl}(3N)$ that appear in (7.1).

The spectrum of $\hat{R}_{\alpha i}^2$ may be obtained from (4.22) by setting $n = 3N$ and applying the formula to the above states (7.2). One finds that for states of these four types the factor $(m_{0,n+1} + \dots + m_{n,n+1}) - (m_{1,n} + \dots + m_{n,n})$ takes the values 2, 1, 1 and 0, respectively, while the second factor $(m_{1,k} + \dots + m_{k,k}) - (m_{1,k-1} + \dots + m_{k-1,k-1})$ lies in the sets $\{0, 1\}$, $\{0, 1\}$, $\{0, 1, 2\}$ and $\{0, 1, 2\}$, respectively. It follows that for each of the possible energy levels E_q the eigenvalues of $\hat{R}_{\alpha i}^2$ are given by

$$R_{\alpha i}^2 = \frac{\hbar r_{\alpha i}^2}{(3N-1)m\omega} \quad \text{with} \quad r_{\alpha i}^2 = \begin{cases} 2, 3 & \text{for } q = 0; \\ 1, 2, 3 & \text{for } q = 1; \\ 0, 1, 2 & \text{for } q = 2. \end{cases} \quad (7.5)$$

Note that, unlike all the $V(p)$ cases with $p > 0$, there exists a configuration with $r_{\alpha i}^2 = 0$, namely one of the $q = 2$ ground state configurations.

To determine the angular momentum content of these states, we can extend (7.1) by considering the chain of subalgebras:

$$\mathfrak{gl}(3N) \rightarrow \mathfrak{gl}(3) \oplus \mathfrak{gl}(N) \rightarrow \mathfrak{so}(3) \oplus \mathfrak{gl}(N) \rightarrow \mathfrak{so}(3). \quad (7.6)$$

The corresponding branchings take the form:

$$\begin{aligned} V_{\mathfrak{gl}(3N)}^{\{1\}} &\rightarrow V_{\mathfrak{gl}(3)}^{\{1\}} \times V_{\mathfrak{gl}(N)}^{\{1\}} \\ &\rightarrow V_{\mathfrak{so}(3)}^1 \times V_{\mathfrak{gl}(N)}^{\{1\}} \\ &\rightarrow N V_{\mathfrak{so}(3)}^1 \end{aligned} \quad (7.7)$$

$$\begin{aligned} V_{\mathfrak{gl}(3N)}^{\{2\}} &\rightarrow V_{\mathfrak{gl}(3)}^{\{2\}} \times V_{\mathfrak{gl}(N)}^{\{2\}} + V_{\mathfrak{gl}(3)}^{\{1,1\}} \times V_{\mathfrak{gl}(N)}^{\{1,1\}} \\ &\rightarrow (V_{\mathfrak{so}(3)}^2 + V_{\mathfrak{so}(3)}^0) \times V_{\mathfrak{gl}(N)}^{\{2\}} + V_{\mathfrak{so}(3)}^1 \times V_{\mathfrak{gl}(N)}^{\{1,1\}} \\ &\rightarrow \frac{1}{2} N(N+1) V_{\mathfrak{so}(3)}^2 + \frac{1}{2} N(N-1) V_{\mathfrak{so}(3)}^1 + \frac{1}{2} N(N+1) V_{\mathfrak{so}(3)}^0 \end{aligned} \quad (7.8)$$

$$\begin{aligned} V_{\mathfrak{gl}(3N)}^{\{1,1\}} &\rightarrow V_{\mathfrak{gl}(3)}^{\{2\}} \times V_{\mathfrak{gl}(N)}^{\{1,1\}} + V_{\mathfrak{gl}(3)}^{\{1,1\}} \times V_{\mathfrak{gl}(N)}^{\{2\}} \\ &\rightarrow (V_{\mathfrak{so}(3)}^2 + V_{\mathfrak{so}(3)}^0) \times V_{\mathfrak{gl}(N)}^{\{1,1\}} + V_{\mathfrak{so}(3)}^1 \times V_{\mathfrak{gl}(N)}^{\{2\}} \\ &\rightarrow \frac{1}{2} N(N-1) V_{\mathfrak{so}(3)}^2 + \frac{1}{2} N(N+1) V_{\mathfrak{so}(3)}^1 + \frac{1}{2} N(N-1) V_{\mathfrak{so}(3)}^0 \end{aligned} \quad (7.9)$$

$$\begin{aligned} V_{\mathfrak{gl}(3N)}^{\{2,1\}} &\rightarrow V_{\mathfrak{gl}(3)}^{\{3\}} \times V_{\mathfrak{gl}(N)}^{\{2,1\}} + V_{\mathfrak{gl}(3)}^{\{2,1\}} \times (V_{\mathfrak{gl}(N)}^{\{3\}} + V_{\mathfrak{gl}(N)}^{\{2,1\}} + V_{\mathfrak{gl}(N)}^{\{1,1,1\}}) + V_{\mathfrak{gl}(3)}^{\{1,1,1\}} \times V_{\mathfrak{gl}(N)}^{\{2,1\}} \\ &\rightarrow (V_{\mathfrak{so}(3)}^3 + V_{\mathfrak{so}(3)}^1) \times V_{\mathfrak{gl}(N)}^{\{2,1\}} + (V_{\mathfrak{so}(3)}^2 + V_{\mathfrak{so}(3)}^1) \times (V_{\mathfrak{gl}(N)}^{\{3\}} + V_{\mathfrak{gl}(N)}^{\{2,1\}} + V_{\mathfrak{gl}(N)}^{\{1,1,1\}}) \\ &\quad + V_{\mathfrak{so}(3)}^0 \times V_{\mathfrak{gl}(N)}^{\{2,1\}} \\ &\rightarrow \frac{1}{3} N(N^2-1) V_{\mathfrak{so}(3)}^3 + \frac{1}{3} N(2N^2+1) V_{\mathfrak{so}(3)}^2 + N^3 V_{\mathfrak{so}(3)}^1 + \frac{1}{3} N(N^2-1) V_{\mathfrak{so}(3)}^0 \end{aligned} \quad (7.10)$$

Combining these results with (7.1) we can write down the generating function

$$\begin{aligned} G(Q, J)_N &= \sum_{q,j=0}^{\infty} n_{q,j}(N) Q^q J^j \\ &= N J Q^0 + N^2 (1 + J + J^2) Q^1 \\ &\quad + \left(\frac{1}{3} N(N^2-1) + N^3 J + \frac{1}{3} N(2N^2+1) J^2 + \frac{1}{3} N(N^2-1) J^3 \right) Q^2 \end{aligned} \quad (7.11)$$

where the coefficient $n_{q,j}(N)$ gives the multiplicity of states of angular momentum j and energy E_q as a function of N . To recover the degeneracy of each energy level one replaces each J^j by the

corresponding dimension $2j + 1$ of $V_{\mathfrak{so}(3)}^j$ to give

$$G(Q)_N = 3N Q^0 + 9N^2 Q^1 + N(9N^2 - 1) Q^2, \quad (7.12)$$

in agreement with (7.4). Similarly, the full angular momentum content for specific values of N may be obtained from (7.11) by setting $Q = 1$. For example, we obtain

$$G(1, J)_{N=1} = 1 + 3J + 2J^2; \quad (7.13)$$

$$G(2, J)_{N=2} = 6 + 14J + 10J^2 + 2J^3. \quad (7.14)$$

Although this example is not very remarkable, it illustrates how the methods constructed here can be used, and it shows that the material developed in this paper can in principle be applied to any unitary representation or class of unitary representations.

8 Conclusions

The theory of WQOs has a deep connection with representation theory of Lie superalgebras. Solutions of WQO systems can be considered not only for the Lie superalgebra $\mathfrak{sl}(1|n)$, but also for other Lie superalgebras. As typical examples, we mention the $\mathfrak{osp}(3|2)$ WQO [28] and the recently studied $\mathfrak{sl}(3|N)$ WQO [29]. The physical properties of the WQO depend on the Lie superalgebra, and on the class of representations considered.

This paper has made a contribution to the study of physical properties of WQOs of type $\mathfrak{sl}(1|n)$, initiated in [5] and performed in detail for the 3-dimensional WQO in [7]-[9]. First of all, our current approach is slightly more general by considering the D -dimensional N -particle WQO of type $\mathfrak{sl}(1|n)$. Secondly, in earlier papers only the solutions corresponding to Fock representations $W(p)$ were investigated. Here, we have initiated the study of solutions related to all (unitary) representations of $\mathfrak{sl}(1|DN)$. This requires the construction of a proper basis (the GZ-basis) for these representations, with explicit actions of the $\mathfrak{sl}(1|DN)$ generators, as given here in Section 2.

In order to emphasize the difference with the previously considered Fock spaces $W(p)$, we have paid particular attention to another special class of representations $V(p)$. These representations are, once again, weight multiplicity free, so they are easy to describe explicitly. The energy spectrum and the spectra of position and momentum operators in these representations have been obtained without much effort. Among the special features we mention: there are only two different energy levels (the difference being $\hbar\omega$); the position operator components are non-commuting operators and have a discrete spectrum, leading to spatial properties that are quite different from those of a canonical quantum oscillator.

In addition, for these representations $V(p)$ the complete angular momentum content has been obtained in the form of a generating function using powerful group theoretical methods.

Another particularly nice result is that, for those solutions related to $V(p)$, a classical limit has been obtained, showing that the behavior of the WQO solution reduces to a classical solution of the harmonic oscillator. For the previously considered Fock space solutions based on $W(p)$, such a limit does not exist.

Wigner Quantum Oscillators, belonging to the class of non-canonical quantum systems, possess many attractive and fascinating properties, both for mathematicians and physicists. We believe that the correspondence principle observed here makes them even more interesting for physics.

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